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Self-avoiding surfaces with knotted boundaries

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Abstract. We consider orientable self-avoiding surfaces, with genus g, embedded in the hypercubic lattice. We consider a surface which has a boundary component and non-zero genus, and devise a construction which will reduce the genus of the surface. This result enables us to study embeddings of surfaces in the three-dimensional lattice, where a surface of genus g may have a boundary component which is a knot of genus $g' \leq g$. We prove that the growth constants of these surfaces are independent of the knot type of the boundary component, and we derive inequalities between the associated critical exponents.

1. Introduction

There is an extensive literature on the properties of self-avoiding surfaces (Durhuus *et al* 1983,1985, Fröhlich 1980, Glaus 1986,1988, Glaus and Einstein 1987). We previously obtained (Janse van Rensburg and Whittington 1989, hereafter referred to as 'I') some rigorous inequalities between the numbers of embeddings of surfaces with a fixed number of boundary components in the lattice $\mathcal{Z}^d$ using the theory of subadditive functions (Hille 1948) as developed by Hammersley (1962) and Wilker and Whittington (1979). The results in I and in this paper are one possible generalisation of polyominoes (Eden 1961, Read 1962, Klarner 1967, Klarner and Rivest 1973) to higher dimensions, where interesting topological properties must be taken into account.

Eguchi and Kawai (1982) considered the number of embeddings of a certain class of surfaces in $\mathcal{Z}^d$ in relation to large-N U(N) gauge theory. They argue that the growth constant of the number of embeddings of these surfaces with fixed genus, and a boundary fixed in space, is independent of its genus. In this paper we consider a similar problem for the numbers of embeddings of self-avoiding surfaces with fixed genus and one boundary component: we prove that the growth constant is independent of the genus.

We start by recalling the results in I. If $s_n(h)$ is the number of self-avoiding surfaces with h boundary components and n plaquettes, then we proved in I that

$$\lim_{n\to\infty} s_n(h)^{1/n} = \beta_h \tag{1.1}$$

exists for $h \geq 0$ in $d \geq 3$ dimensions and for $h \geq 1$ in $d = 2$ dimensions. Furthermore, if $h \geq 1$, then β_h is independent of h. We defined a critical exponent ϕ_h by assuming that $s_n(h) \sim C_h n^{-\phi_h} \beta_h^n$, and proved that

$$\left.\begin{aligned} &\phi_h \geq \phi_{h+1} \\ &\phi_1 - h \geq \phi_{h+1} \geq \phi_1 - (3/2)h \end{aligned}\right\} \quad \forall h \geq 1 \quad \text{and} \quad d = 2 \tag{1.2}$$

and

$$\phi_h \geq \phi_{h+1} \geq \phi_1 - 2h \qquad \forall h \geq 1 \quad \text{and} \quad d \geq 3. \tag{1.3}$$

These results are reminiscent of results obtained for the c model in lattice animals (Soteros and Whittington 1988, Madras *et al* 1989). We also conjectured that $\phi_h = \phi_1 + 1 - h$, reflecting a similar result in lattice animals. This model of surfaces behaves similarly to lattice animals, with c, the cyclomatic index of the animals, replaced by h, the number of boundary components of the surfaces. In this paper, we consider the situation where h is replaced by g (the genus of the self-avoiding surface), and show that self-avoiding surfaces with one boundary component and genus g behave rather like lattice animals with the cyclomatic index replaced by the genus.

The incidence of knots in lattice polygons and the enumeration of knots in S^3 by crossings have also received much attention in the literature (Michels and Wiegel 1986, Sumners 1987, Ernst and Sumners 1987, Sumners and Whittington 1988, Pippenger 1989). As yet, there is no rigorous proof that a lattice polygon of a fixed knot-type has a growth constant independent of the knot type, although this is believed to be true. In three dimensions there is the possibility that a boundary component of a surface could be a knot, depending on the embedding of the surface (think for example of the Seifert surface of a knot). In figure 1 we have two embeddings of a torus with one boundary component. In figure 1(a) it has the unknot as boundary, and in figure 1(b) the trefoil. In general, an embedding of a torus with g handles can have any knot with genus less or equal to g as a boundary component. We shall show that once punctured surfaces have a growth constant independent of the knot type of the boundary.

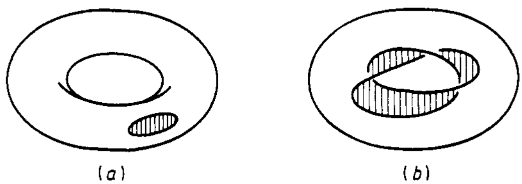

Figure 1. Two embeddings of the 1-torus in $\mathcal{R}^3$. (a) has the unknot as boundary and (b) has the trefoil knot as boundary.

Let $\mathcal{Z}^d$ be the d-dimensional hypercubic lattice where $d \geq 3$. A ***plaquette*** is the interior and boundary of a unit square whose vertices are in $\mathcal{Z}^d$. Two plaquettes are *joined* if they share a common edge, and two plaquettes are *connected* if they are elements in a sequence of plaquettes such that neighbouring pairs are joined. A surface is a collection of connected plaquettes. We call a vertex on a surface *common* if the plaquettes incident on it form a connected set. A surface is *self-avoiding* if every edge on the surface is incident on at most two plaquettes, and if all the vertices in the surface are common. In the rest of this paper we shall mean *self-avoiding surface* whenever we say *surface*. Edges incident on only one plaquette form the boundary of the surface, which consists in general of several disjoint components (each of which is a polygon in $\mathcal{Z}^d$).

This paper is organised in the following way. In section 2 we consider $\mathcal{S}_n(h,g)$, the set of *orientable* surfaces with h boundary components and genus g, consisting of n plaquettes in $\mathcal{Z}^d$. Let the cardinality of $\mathcal{S}_n(h,g)$ be $s_n(h,g)$. We briefly consider concatenation of these surfaces, and then focus our attention on surfaces in the set

$\mathcal{S}_n(1,g)$, where there is only one boundary component. We prove that it is possible to reduce the genus of the surface by removing a strip of plaquettes from it. This construction immediately implies the existence of the limit

$$\lim_{n\to\infty} s_n(1,g)^{1/n} = \beta(1,g) \tag{1.4}$$

where $\beta(1,g)$ is independent of g for all $g \geq 0$. Assuming that there exist constants C_g and exponents $\phi(1,g)$ such that $s_n(1,g) \sim C_g n^{-\phi(1,g)}\beta(1,g)^n$, we then derive the following relation among the exponents $\phi(1,g)$:

$$\phi(1,g-1) \geq \phi(1,g) \geq \phi(1,0) - 4g \tag{1.5}$$

which is reminiscent of equation (1.3).

In section 3 we shift our attention to three dimensions. We consider the set $\Sigma_n(T_g, g')$ of all orientable surfaces embedded in $\mathcal{Z}^3$ consisting of n plaquettes, with genus g', and having a single boundary component which is a knot T_g, with genus $g \leq g'$. Let the cardinality of $\Sigma_n(T_g, g')$ be $s_n(T_g, g')$. We apply the construction in section 2 to this set and prove that the limit

$$\lim_{n\to\infty} s_n(T_g, g')^{1/n} = \beta(T_g, g') \tag{1.6}$$

exists, and is independent of T_g and g'. Assuming that there exist constants $C_{g'}(T_g)$ and exponents $\phi(T_g, g')$ such that $s_n(T_g, g') \sim C_{g'}(T_g)n^{-\phi(T_g,g')}\beta(T_g, g')^n$, we can derive some relations among the $\phi(T_g, g')$. In particular, if we define $\phi(T_g) = \phi(T_g, g)$ (for surfaces with genus equal to the genus of the knotted boundary), then

$$\phi(\emptyset) + 2 \geq \phi(T_g) \geq \phi(\emptyset) - 4g \tag{1.7}$$

where $\emptyset$ is the unknot, and T_g is any (prime or compound) knot. Relating $\phi(T_g)$ and $\phi(T'_{g'})$ to each other is more difficult.

Let T be a knot which meets a plane E in $\mathcal{R}^3$ in exactly two points P and Q. The arc of T from P to Q can be closed by an arc in E to obtain a knot T_1. The other arc (from Q to P) is closed in a similar way to get a knot T_2. The knot T is called the *product* of T_1 and T_2, and we denote it by $T_1 \# T_2$. T is called a *compound* knot with *factors* T_1 and T_2 if neither T_1 nor T_2 are the unknot (Burde and Zieschang 1985). Suppose that $T'_{g'}$ is a compound knot containing the knot T_g (i.e., there is a knot $T''_{g''}$ such that $T'_{g'} = T_g \# T''_{g''}$ and $g' = g + g''$). Then

$$\phi(T_g) + 2 \geq \phi(T'_{g'}). \tag{1.8}$$

We are not able to say more than this. For example, we would like to relate $\phi(T_g)$ and $\phi(T'_{g'})$ for any knots (prime or compound). Moreover, if $g = g'$ (so that the knots are different, but have the same genus), what is the relation then? So far, our constructions do not provide any clues.

We conclude the paper with a few remarks in section 4. We discuss several problems closely related to this paper and consider several unsolved problems in this area.

2. Surfaces in $\mathcal{Z}^d$

Let $\mathcal{S}_n(h,g)$ be the set of orientable surfaces consisting of n plaquettes, with h boundary components and genus g, in $\mathcal{Z}^d$, $d \geq 3$. Let the cardinality of $\mathcal{S}_n(h,g)$ be $s_n(h,g)$. The concatenation of surfaces and the stripping of plaquettes between different boundary components were considered in detail in I. Therefore we state the following results without proof.

Lemma 2.1. Let $d \geq 3$. Then $s_n(h,g)$ obeys the following inequalities.
(i) $s_n(h,g) \leq s_{n+4}(h,g)$.
(ii) $s_n(h,g) \leq s_{n+C}(h,g+1)$ where C is a positive constant.
(iii) $s_n(h_1,g_1)s_m(h_2,g_2) \leq s_{n+m+\lambda}(h_1+h_2,g_1+g_2)$ where λ is a positive constant.
(iv) $s_n(h,g) \leq n(\lceil \frac{n}{4} \rceil - 1)s_{n+i_n}(h-1,g)$ where i_n is an integer in the interval $[0,3]$.

We shall use these results widely in this paper. Note that lemma 2.1 is not only applicable to the cardinality of $\mathcal{S}_n(h,g)$, but also to other sets of surfaces, providing that we fix the number of boundary components. If we sum over the number of boundary components, then lemma 2.1(i), (ii), and (iii) are still true, with the argument h summed over.

Let $\sigma \in \mathcal{S}_n(h,g)$. The rank ρ of the first homology group $\mathcal{H}_1$ of σ (the first Betti number) is given by e.g. (Kaufman 1983)

$$\rho = h - 1 + 2g. \tag{2.1}$$

If $h = 1$ then $\rho = 2g$. (If the surface has only one boundary component, then the first homology group is free Abelian of rank $2g$.) We shall now design a construction which will reduce the rank of $\mathcal{H}_1$ of σ for $h = 1$, giving us an inequality between $s_n(1,g)$ and $s_n(1,g-1)$.

Lemma 2.2. Let $d \geq 3$ and $\sigma \in \mathcal{S}_n(1,g)$ where $g \geq 1$. Then the genus of σ can be reduced by 1 by removing at most $\lceil n/2 \rceil$ or adding at most $(g-1)C$ (where C is a fixed, positive integer, independent of n) plaquettes to σ, creating a new boundary component in the process. Moreover, we find that

$$\begin{aligned} s_n(1,g) &\leq gn\left(\lceil \frac{n}{2} \rceil - 1\right) s_{n+j_n}(2,g-1) \\ &\leq gn\left(\lceil \frac{n}{2} \rceil - 1\right)(n+(g-1)C+3)\left(\lceil \frac{n+(g-1)C+3}{2} \rceil - 1\right) \\ &\quad \times s_{n+i_n}(1,g-1) \end{aligned}$$

where $(g-1)C \leq j_n \leq (g-1)C+3$ and $(g-1)C+3 \leq i_n \leq (g-1)C+6$.

Proof. Let $\sigma \in \mathcal{S}_n(1,g)$. Every orientable 2-manifold with one boundary component is homeomorphic to a torus with g handles and with an open disc removed. Choose a base point b_1, and a second point b_2, on the boundary of σ. Let $\{\mu_i\}_{i \in I}$ be the set of all 1-cycles on σ, labelled by the (infinite) index set I with the following properties.

(1) None of the μ_i are null-homologous.

(2) Each μ_i contains the points b_1 and b_2, where the segment $b_2 \to b_1$ runs on the boundary of σ.

(3) If we consider each μ_i to consist of the segments $b_1 \to b_2$ and $b_2 \to b_1$, then $b_1 \to b_2$ passes through the midpoints of each plaquette that it visits and passes from one plaquette to the next only through common edges.

This setup is illustrated schematically in figure 2 for a punctured torus. The broken lines represent two 1-cycles which have the properties (1)–(3) above. Let $d(\mu_i)$, the length of the 1-cycle μ_i, be the number of midpoints of plaquettes that the segment $b_1 \to b_2$ visits on σ. Vary $i \in I$, and b_1 and b_2 on the boundary of σ to find that 1-cycle with the properties above such that

$$d(\mu_{\min}) \leq d(\mu_i) \qquad \forall i \in I \quad \forall b_1 \quad \forall b_2.$$

Thus, $\mu_{\min}$ is that 1-cycle on σ, starting on the boundary and terminating there, which visits the least number of plaquettes and is not null-homologous.

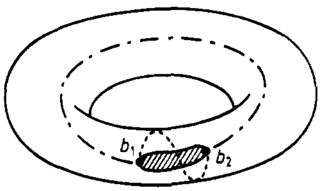

Figure 2. An orientable 2-manifold with one boundary component and genus 1. Two 1-cycles with the properties in lemma 2.2 are indicated. Each cycle consists of two segments, the first segment runs from b_1 to b_2 (indicated by the broken lines), and the second is from b_2 to b_1 and runs along the boundary component of the surface.

Delete every plaquette on σ whose midpoint is visited by $\mu_{\min}$, transforming σ into σ'. Suppose that we delete j plaquettes in this way. Since b_1 and b_2 are both on the boundary of σ, and $\mu_{\min}$ is not null-homologous, we do not disconnect σ, but we 'cut' through a 'handle' of the torus in figure 2, reducing the rank of the first homology group by at least 1. An easy application of the Jordan–Brouwer curve theorem (Greenberg and Harper 1981) shows that we now have two boundary components (σ is orientable). Hence, by equation (2.1) the genus of σ' is $g' \leq g - 1$. This inequality may be strict, as we show in figure 3.

To see that $1 \leq j \leq \lceil n/2 \rceil$, suppose that $j > \lceil n/2 \rceil$. By the choice of $\mu_{\min}$, j is the minimum number of plaquettes that we must remove to reduce the genus of σ by at least 1. Let this set of plaquettes be L. Since $\mathcal{Z}^d$ has girth 4, there is at least one strip of plaquettes, L', adjacent to L, which we can remove instead to reduce the genus, containing at least j plaquettes. Therefore, L and L' have at least $2j > n$ plaquettes. This is a contradiction, so $j \leq \lceil n/2 \rceil$. This deletion of plaquettes is therefore a map

$$\Lambda : \sigma \to \sigma' \in \mathcal{S}_{n-j}(2, g')$$

where $\lceil n/2 \rceil \geq j \geq 1$ and $0 \leq g' \leq (g-1)$. To see that this map is at most n to 1 we argue as follows. Suppose that we remove a strip L consisting of j plaquettes to transform σ into σ'. The maximum number of edges on the boundary of σ' is $2(n-j+1)$. Every plaquette in L, except (perhaps) for the first and last, must have two edges on the boundary of σ'. Therefore, the maximum number of ways that we

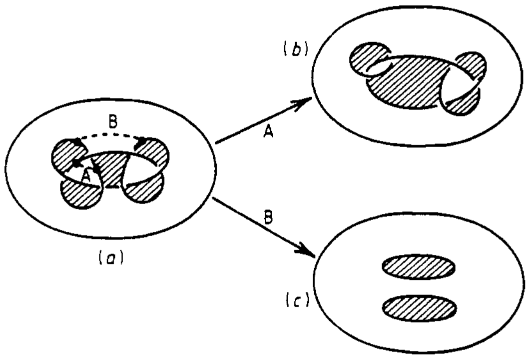

Figure 3. Removing a strip of plaquettes along A reduces the rank of the first homology group by one, resulting in figure 3(b). If we remove it along B instead, then it reduces the rank of the first homology group by three and we find figure 3(c).

can put back the connected strip L into σ' to get σ is bounded above by $2(n-j+1)/2$. This is a maximum if $j = 1$. Therefore

$$s_n(1,g) \leq n \sum_{j=(n-\lceil \frac{n}{2} \rceil)}^{n-1} \sum_{g'=0}^{g-1} s_j(2,g').$$

Apply lemma 2.1(ii) $(g-1-g')$ times to $s_j(2,g')$. This gives

$$s_n(1,g) \leq n \sum_{j=(n-\lceil \frac{n}{2} \rceil)}^{n-1} \sum_{g'=0}^{g-1} s_{j+(g-1)C-g'C}(2,g-1).$$

We can apply lemma 2.1(i) to each term in the sum above. Since we add plaquettes in groups of four, each term is bounded by $s_{n+z_n}(2,g-1)$, where z_n is an integer in the interval $[(g-1)C,(g-1)C+3]$ depending on $(j+(g-1)C-g'C) \bmod 4$. Let y_n be the integer in the interval $[(g-1)C,(g-1)C+3]$ such that

$$s_{n+y_n}(2,g-1) = \max_{(g-1)C \leq j \leq (g-1)C+3} s_{n+j}(2,g-1).$$

Then each term above is bounded by $s_{n+y_n}(2,g-1)$. We evaluate the sums to find

$$s_n(1,g) \leq gn(\lceil \frac{n}{2} \rceil - 1)s_{n+y_n}(2,g-1)$$

where $(g-1)C \leq y_n \leq (g-1)C+3$. Lastly, apply lemma 2.1(iv) to this result.

We are now in the position to consider equation (1.4). In I we proved that a surface with h boundary components has properties very similar to a lattice animal with cyclomatic index h. We consider a different situation here: a surface with a single boundary component and genus g. In theorem 2.3 we see that the 'handles' on the surface are like cycles on the animal—the growth constants are independent of the genus. In addition to this, if we assume the existence of a critical exponent then we can derive inequalities relating the exponents which are very similar to existing relations for lattice animals.

Theorem 2.3. Let $d \geq 3$. Then:

(i) there exists a positive number $\beta(1,0)$, dependent only on d, such that

$$\lim_{n\to\infty} s_n(1,0)^{1/n} = \beta(1,0)$$

(ii) and there exist positive numbers $\beta(1,g)$ such that

$$\lim_{n\to\infty} s_n(1,g)^{1/n} = \beta(1,g)$$

and where $\beta(1,g) = \beta(1,0)\ \forall g$.

(iii) Suppose that for every $g \geq 0$ there exists a constant C_g such that

$$s_n(1,g) \sim C_g n^{-\phi(1,g)} \beta(1,g)^n .$$

Then the exponents $\phi(1,g)$ are related to each other via

$$\phi(1,g-1) \geq \phi(1,g) \geq \phi(1,0) - 4g.$$

Proof. (i) Janse van Rensburg and Whittington (1989).

(ii) By lemma 2.1(iii) and lemma 2.2:

$$s_{n-m}(1,g-1)s_{m-\lambda}(0,1) \leq s_n(1,g) \leq \mathrm{O}(gn^4)s_{n+i_n}(1,g-1)$$

where m is chosen such that $s_{m-\lambda}(0,1) > 0$ and $(g-1)C + 3 \leq i_n \leq (g-1)C + 6$. Take the $1/n$ power and let n go to infinity. The existence of the limit then follows from an inductive argument on (i) above.

(iii) Apply lemma 2.2 g times to the last inequality above, substitute the assumption, divide by $\beta(1,0)^n$, take logs, divide by $\log n$ and let $n \to \infty$. This gives the string of inequalities.

3. Surfaces with knotted boundary components

In this section we restrict our attention to three dimensions, and consider surfaces with a single boundary component which is a (prime or compound) knot of genus g. Let the set of orientable surfaces with genus g' consisting of n plaquettes with a boundary component being the knot T_g (with genus g) be $\Sigma_n(T_g, g')$, where $g' \geq g$. Let the cardinality of $\Sigma_n(T_g, g')$ be $s_n(T_g, g')$. We now prove the existence of the limit in equation (1.5).

Theorem 3.1. Let $d = 3$. Then there exists a positive number $\beta(T_g, g')$ such that

$$\lim_{n\to\infty} s_n(T_g, g')^{1/n} = \beta(T_g, g')$$

and $\beta(T_g, g') = \beta(\emptyset, 0)$ where $\emptyset$ is the unknot.

Proof. Let $s_n(T_g, T'_{g'}, g'')$ be the number of surfaces with genus g'' consisting of n plaquettes and having two boundary components of knot types T_g and $T'_{g'}$ (where

$g+g' \leq g''$). Then the result follows from lemma 2.1(iii) and (iv) and lemma 2.2. We find

$$\begin{aligned} s_{n-m}(\emptyset,0)s_{m-\lambda}(T_g,g') &\leq s_n(T_g,\emptyset,g') \\ &\leq O(n^2)s_{n+i_n}(T_g,g') \\ &\leq O(n^{4g'+2})s_{n+k_n}(\emptyset,0) \end{aligned}$$

where m is fixed such that $s_{m-\lambda}(T_g,g') > 0$ and where $0 \leq i_n \leq 3$ and $(g'-1)(3+g'C/2) \leq k_n \leq (g'-1)(6+g'C/2)$. Take the $1/n$ power and let n go to infinity. The existence of the limit follows by induction on theorem 2.3(i).

Theorem 3.1 implies that $s_n(T_g,g') = \beta(T_g,g')^{n+o(n)}$. As in theorem 2.3, suppose that there exist constants $C_{g'}(T_g)$ and exponents $\phi(T_g,g')$ such that

$$s_n(T_g,g') \sim C_{g'}(T_g)n^{-\phi(T_g,g')}\beta(T_g,g')^n \tag{3.1}$$

where $\phi(T_g,g')$ is an exponent depending on the genus and the knot type of the boundary. If we are interested in the relations between $\phi(T_g,g'-1)$ and $\phi(T_g,g')$, where $g'-1 \geq g$, then lemma 2.1(ii) and lemma 2.2 applied g' times give the series of inequalities

$$s_{n-C}(T_g,g'-1) \leq s_n(T_g,g') \leq O(n^{4g'})s_{n+i_n}(\emptyset,0) \tag{3.2}$$

where $\emptyset$ is the unknot, $(g'-1)(3+g'C/2) \leq i_n \leq (g'-1)(6+g'C/2)$, and where C is the constant in lemma 2.1(ii). Hence

$$\phi(T_g,g'-1) \geq \phi(T_g,g') \geq \phi(\emptyset,0) - 4g'. \tag{3.3}$$

This result does not relate the exponents of surfaces with different knot types as boundary components to each other. Consider the exponent $\phi(T_g) = \phi(T_g,g)$. Lemma 2.1(iii) and (iv) and lemma 2.2 imply that

$$\begin{aligned} s_{n-m}(\emptyset,0)s_{m-\lambda}(T_g,g) &\leq s_n(\emptyset,T_g,g) \\ &\leq O(n^2)s_{n+i_n}(T_g,g) \\ &\leq O(n^{4g+2})s_{n+j_n}(\emptyset,0) \end{aligned} \tag{3.4}$$

where m is fixed such that $s_{m-\lambda}(T_g,g) > 0$, and $0 \leq i_n \leq 3$ and $(g-1)(3+gC/2) \leq j_n \leq (g-1)(6+gC/2)$. Substituting (3.1), we find that

$$\phi(\emptyset) + 2 \geq \phi(T_g) \geq \phi(\emptyset) - 4g. \tag{3.5}$$

We can also relate $\phi(T_g)$ and $\phi(T'_{g'})$ if $T'_{g'}$ is a compound knot containing T_g (prime or compound). Then there exists a knot $T''_{g''}$ such that $T'_{g'} = T_g \# T''_{g''}$. By lemma 2.1(iii) and (iv) we have

$$\begin{aligned} s_{n-m}(T_g,g)s_{m-\lambda}(T''_{g''},g'') &\leq s_n(T_g,T''_{g''},g+g'') \\ &\leq O(n^2)s_n(T'_{g'},g') \end{aligned} \tag{3.6}$$

where m is fixed such that $s_{m-\lambda}(T''_{g''}, g'') > 0$. Substitute (3.1), then we find

$$\phi(T_g) + 2 \geq \phi(T'_{g'}). \tag{3.7}$$

We state these results together in the following theorem.

Theorem 3.2. Let $d = 3$. Suppose that there exists a constant $C_{g'}(T_g)$ and an exponent $\phi(T_g, g')$ such that equation (3.1) is true. Then the exponents obey the following relations.

(i) Let T_g be any knot and suppose that $g' > 0$. Then

$$\phi(T_g, g' - 1) \geq \phi(T_g, g') \geq \phi(\emptyset, 0) - 4g'$$

where $\emptyset$ is the unknot.

(ii) Suppose that T_g is any knot. Then

$$\phi(\emptyset) + 2 \geq \phi(T_g) \geq \phi(\emptyset) - 4g$$

where $\phi(T_g) = \phi(T_g, g)$.

(iii) Suppose that there exists a knot $T''_{g''}$ such that $T'_{g'} = T_g \# T''_{g''}$ and $g' = g + g''$. Then

$$\phi(T_g) + 2 \geq \phi(T'_{g'}).$$

4. Discussion

(1) We considered the existence of the limit in (1.4) in theorem 2.3. It is now an easy task to combine the results in lemmas 2.1(iii) and (iv) and lemma 2.2 to prove the existence of the limit

$$\lim_{n \to \infty} s_n(h, g)^{1/n} = \beta(h, g) \tag{4.1}$$

and to show that $\beta(h, g) = \beta(1, 0)$ for all $h \geq 1$ and $g \geq 0$. Similarly, we can use the inequalities to consider relations among critical exponents in exactly the same fashion as in theorem 2.3(iii). We find that

$$\phi(h - \nu, g - \mu) \geq \phi(h, g) \geq \phi(1, 0) - 2(h - 1) - 4g \tag{4.2}$$

where $\nu = 0$ or 1 and $\mu = 0$ or 1.

(2) In theorem 3.2 we state some relations between the exponents of a surface with boundary component of a fixed knot type. This list is woefully incomplete. In particular, we have no result which indicates a relation between the exponents of two prime knots with different genus. Moreover, we would like to be able to relate the exponents of every given pair of knots. There is an obvious question here: if two knots have the same genus, will they have the same exponent, or will the exponent be determined by some other knot invariant? This is an interesting question, and deserves further investigation.

(3) We had some success in studying the growth constants of these surfaces, but some questions remain unanswered. In particular, if we define $s_n(1) = \sum_{g=0}^{\infty} s_n(1,g)$, then it is easily seen that the limit $\lim_{n\to\infty} s_n(1)^{1/n} = \beta(1)$ exists and that $\beta(1) \geq \beta(1,0)$. Is this inequality strict? If it is, then we know that exponentially few surfaces have fixed genus. In 1985 J Fröhlich (1985) posed an interesting question closely related to this point. Is there a regime where an interface (between two phases in for example the Ising model) has non-zero genus with probability 1 in the scaling limit? Consider $s_n(0,g)$, the number of closed surfaces with genus g. Let $s_n(0) = \sum_{g=0}^{\infty} s_n(0,g)$. Can we prove that $\lim_{n\to\infty} s_n(0,g)^{1/n} = \beta(0,g)$ exists for all g? It obviously exists for $g = 0$. If $\beta(0,0) < \max_g \beta(0,g)$, or if $\beta(0,0) < \beta(0)$, where $\lim_{n\to\infty} s_n(0)^{1/n} = \beta(0)$, then there is a regime where the interface has non-zero genus in the scaling limit.

(4) Closely related to the situation in (3) is the following. Let $T^m = (T\#T\#\cdots\#T)$ be a compound knot consisting of m copies of the knot T. Let $s_n(T) = \sum_{m=0}^{\infty} s_n(T^m, mg)$, if we assume that T has genus g. By lemmas 2.1(iii) and (iv) the limit $\lim_{n\to\infty} s_n(T)^{1/n} = \tau_T$ exists. Similarly, let $s_n(C)$ be the number of embeddings of surfaces with boundary any compound knot. Then by lemma 2.1(iii) and (iv), $\lim_{n\to\infty} s_n(C)^{1/n} = \tau_C$ exists. If $s_n(P)$ is the number of embeddings with any prime knot as boundary, then our methods fail to prove that $\lim_{n\to\infty} s_n(P)^{1/n} = \tau_P$ exists. Suppose that it does exist. How are τ_T, τ_C and τ_P related to $\beta(\emptyset,0)$ (theorem 3.1)? If τ_T (τ_C and τ_P) $> \beta(\emptyset,0)$, then exponentially few embeddings of surfaces have the unknot as boundary as compared to $s_n(T)$ ($s_n(C)$ and $s_n(P)$). Hence, exponentially few surfaces will have the unknot as boundary, as compared to all the embeddings in $\mathcal{Z}^3$. This is similar to the fact that exponentially few polygons in $\mathcal{Z}^3$ are the unknot (Sumners and Whittington 1988).

(5) It is easy to perform a similar study with surfaces with two boundary components and where we consider the linking of the boundary components. By using lemma 2.1(iv) it is easy to prove that growth constants exist and we can postulate the existence of a critical exponent and derive inequalities relating these exponents for some cases. As in the case of knots, this list will be incomplete, and more work will be necessary to fill it out. Furthermore, given a surface with two boundary components, are the boundary components unlinked in only exponentially few cases?

(6) We did not consider non-orientable surfaces in this paper. It is not difficult to extend the results in this paper to that case as well.

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